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# VALUING EUROPEAN OPTIONS WHEN THE TERMINAL VALUE OF THE UNDERLYING ASSET IS UNOBSERVABLE

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# VALUING EUROPEAN OPTIONS WHEN THE TERMINAL VALUE OF THE UNDERLYING ASSET IS UNOBSERVABLE

## ABSTRACT

This paper addresses a problem which arises when option pricing techniques are applied to the valuation of claims to real assets, where the claim allows its owner to make strategic decisions in the future. Previous research has shown that such claims can be treated as options on the underlying asset. The problem is that, as opposed to the standard assumption in previous academic research in this area, the value of the underlying asset will typically not be perfectly observable at the time of the future (exercise) decision making opportunity. By making specific assumptions on the nature of the asset value unobservability and on the existence of a unique equilibrium pricing measure, we derive a quasi-closed form solution to the value of such a claim. Furthermore, we show that this solution is identical to the Black-Scholes formula with either time to maturity or asset return variance reduced by a factor which decreases in the variance of the noise at maturity. Two examples, treating the valuation of risky corporate debt and of a research and development project, respectively, demonstrate that the issue can be of significant economic importance. Also, the last example demonstrates that, by letting the variance of the asset value noise at maturity be endogenous, we can derive a simple model of the value of information systems to corporate decision makers.



Option pricing models have been applied to solve a large number of problems beyond the valuation of exchange traded puts and calls on common stock, which was the main focus of the seminal papers by Black and Scholes (1973) and Merton (1973). Such research has focused on optimal strategic decision making and the accompanying valuation of the right to decide. Examples of such papers<sup>1</sup> include research on the valuation of natural resources (Brennan and Schwartz, 1985; McDonald and Siegel, 1986; Gibson and Schwartz, 1990; Paddock, Siegel and Smith, 1988), of risky corporate debt (Merton, 1974; Geske 1977; Kan 1991), of undeveloped land (Williams, 1989; Titman, 1985; Quigg, 1990), of research and development limited partnerships (Shevlin, 1991), and of manufacturing flexibility (Myers, 1977; Majd and Pindyck, 1987; Mason and Trigeorgis, 1987; Pindyck, 1988; Hodder and Triantis, 1990). Common to most of these applications is that the exact value of the underlying asset of the option may not be perfectly observable at the time an exercise decision is made. This fact has generally been disregarded by finance researchers, who have typically assumed a structure on the problem so that standard option pricing models apply.

This paper demonstrates that, if instead of being able to observe the true value of the underlying asset at maturity the decision maker (option holder) is only allowed to observe a noisy signal, several aspects of the problem change. In particular, the decision maker now has to solve a non-trivial problem in order to

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<sup>1</sup> This literature is vast and rapidly growing, so a full survey would be outside the scope of this paper. For a good introduction to this area together with an overview of the early work see Mason and Merton (1985).

decide whether to exercise an expiring option. We show that, under these circumstances the option holder will, with a positive probability, exercise out-of-the money option, as well as not exercise in-the-money options. This not only uniformly reduces the time value of the option, it changes a rationally exercised option from being a zero liability asset into an asset offering potentially large losses. Furthermore, we show that there will be a range of values of the underlying asset where the value of a call option will decrease in the true value of the asset, counter to traditional intuition. By making certain assumptions on the exact nature of the noise in the observed value of the underlying asset and on the existence of a unique equilibrium pricing measure, we derive a quasi-closed form solution for the pricing of European call options in this setting. Furthermore, we show that this solution is identical to the Black-Scholes formula for the call value with either return variance or time to maturity reduced by a factor that decreases in the variance of the noise at maturity. The pricing results are interesting from a general point of view, in that they demonstrate that the source of the time value in a European option is a measure of the average "revealed return volatility" until maturity, rather than total return volatility.

The valuation model is applied to rework Merton's (1974) analysis of risky corporate debt, and to investigate a modified version of Shevlin's (1991) valuation of R&D limited partnerships. In both cases we demonstrate that even fairly moderate levels of noise can have a non-trivial economic impact. The latter example also makes it clear that by letting the precision of future information be a decision

variable, a fairly rich model of corporate decision making can arise out of a simple structure. We conclude by pointing out possible venues for generalization and applications of this framework.

## I. THE BASIC MODEL

In order to focus our attention on the specific effects of the asset price being unknown at the maturity of the option, we shall make a number of simplifying assumptions. This will allow us to draw very sharp conclusions, and it also facilitates comparison with previous theoretical models. Therefore, consider an economy with a risk free asset earning interest at a constant rate  $r$ , a risky asset, and a call option on this asset. It is assumed that the value of the risky asset follows a geometric Brownian Motion with a variance of  $\sigma^2$  per time period. At time 0 we can observe its initial value<sup>2</sup>,  $S_0$ , but from time 0 to time  $\tau$ , the maturity date of the option, the true value of the risky asset is unobservable. Instead of observing the true asset value at maturity,  $S_\tau$ , the option holder can observe  $Z = S_\tau e^x$ , where  $x$  is independent of the asset price and assumed to be distributed normally with a mean of  $(-\gamma^2/2)$  and a variance of  $\gamma^2$ . Here,  $e^x$  represents noise, which by assumption is unbiased<sup>3</sup>, multiplicative, and lognormally distributed. Lognormality is assumed for reasons of tractability, whereas we believe that the multiplicative nature of the noise corresponds

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<sup>2</sup> The assumption that  $S_0$  is observable is discussed further in Section III.

<sup>3</sup> The mean of  $x$  is assumed to be  $-\gamma^2/2$  (rather than 0) in order for the mean of  $e^x$  to be 1.

fairly well to the way people estimate values.

To isolate the informational effects of this unobservability we also assume that equilibrium pricing of all contingent claims dependent on  $S_\tau$  can be performed in the standard way by taking expectations under a unique equilibrium risk adjusted probability measure, under which the asset price discounted by the risk free interest rate follows a martingale. We will also assume that there is no risk premium attached to the uncertainty induced by  $x$ . This is reasonable to the extent that  $x$  is pure noise and small relative to aggregate economy, in which case it will be approximately independent of aggregate consumption. However, as our pricing assumptions cannot be justified directly by arbitrage arguments they should be considered as arising from an equilibrium. Such an equilibrium was first derived by Rubinstein (1976), and relies on a representative agent whose preferences can be represented by a von Neumann-Morgenstern utility function featuring constant relative risk aversion. For a recent example of applications of this framework to contingent claims pricing, see the papers of Milne and Turnbull (1989,1991).

The option to be considered is a European call option on the asset with maturity date  $\tau$  and exercise price  $K$ . The first decision to be made is with respect to the rational exercise policy of the option. This is trivial if  $\gamma=0$ , i.e. in the standard case, but not so here. Under the assumptions outlined above, the holder of the option will make the exercise decision on the basis of an estimate of the true value of the underlying asset at maturity, formed from the initial value and the observed noisy signal,  $Z$ . The optimal policy in this setting is to exercise the option if and only

if the conditional expectation of the terminal asset price under the risk adjusted probability measure is greater than the exercise price. We show that the conditional expectation in question is given by a certain geometric variance-weighted average of the time 0 price and the noisy signal, corrected for loss of convexity.

### Proposition 1

The optimal exercise policy at maturity of the option described above is to exercise the option if and only if  $P > K$ , where:

$$P = Z^\alpha F_{0,\tau}^{(1-\alpha)} e^{\left(\frac{\alpha\gamma^2}{2}\right)} \quad (1)$$

Here,  $F_{0,\tau}$  is the time 0 forward price for the underlying asset, i.e.:

$$F_{0,\tau} = S_0 e^{r\tau} \quad (2)$$

and  $\alpha$  is given by:

$$\alpha = \frac{\sigma^2 \tau}{\gamma^2 + \sigma^2 \tau} \quad (3)$$

### Proof

Under the risk adjusted probability measure, option holders act as if they were risk neutral, independent of their true preferences. Thus, it suffices to show that the

policy given in the proposition maximizes the expected cash flow from the option. Since not exercising ensures a cash flow of zero, it therefore follows immediately that it is rational to exercise if and only if the expected cash flow from exercising (under the risk adjusted probability measure, conditional on the noisy signal) is positive.

For  $P$  to be (a version of) the conditional expectation of  $S_\tau$  given  $Z$ , it needs to be measurable with respect to the  $\sigma$ -field generated by  $Z$ ,  $\sigma(Z)$ , and have the property that (see e.g. Shirayev 1984, p. 211):

$$E[\chi_A S_\tau] = E[\chi_A P] \quad , \quad \forall A \in \sigma(Z) \quad (4)$$

Here,  $\chi_A$  is the indicator variable that takes on a value of 1 if the event  $A$  occurs and is zero otherwise.  $P$  is obviously  $\sigma(Z)$ -measurable, since it is an explicit function of  $Z$ . To verify (4) we will use the fact that this function is globally invertible and a.s. positive, and therefore, by a standard limit argument it follows that the set of functions of the form  $\chi_{\{P>K\}}$  for  $K>0$  generates  $\sigma(Z)$ . Thus, we need to confirm that :

$$E[(S_\tau - P)\chi_{\{P>K\}}] = 0 \quad , \quad \forall K \geq 0 \quad (5)$$

If we rewrite  $S_\tau$  as  $e^y$ , use the definition of  $Z$ , and reorganize slightly we can express the left hand side of (5) as:

$$E[(S_\tau - P)\chi_{\{P>K\}}] = E\left[\left(e^y - e^{\frac{\alpha y + \alpha x + \frac{\alpha \gamma^2}{2}}{2}} F_{0,\tau}^{(1-\alpha)}\right) \chi_{\left\{y > \frac{1}{\alpha} \ln(K/F_{0,\tau}) + \ln(F_{0,\tau}) - \frac{\gamma^2}{2} - x\right\}}\right] \quad (6)$$

We will compute this expectation by first conditioning on  $x$  and integrating over the distribution of  $y$ , and subsequently integrate the result over  $x$ :

$$\begin{aligned} & E[(S_\tau - P)\chi_{\{P>K\}}] \\ &= \int_{-\infty}^{\infty} \int_{\frac{1}{\alpha} \ln(K/F_{0,\tau}) + \ln(F_{0,\tau}) - \frac{\gamma^2}{2} - x}^{\infty} \left( e^y - e^{\frac{\alpha y + \alpha x + \frac{\alpha \gamma^2}{2}}{2}} F_{0,\tau}^{(1-\alpha)} \right) \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(y - \ln(F_{0,\tau}) + \frac{\sigma^2\tau}{2})^2}{2\sigma^2\tau}} dy \frac{1}{\sqrt{2\pi\gamma^2}} e^{-\frac{(x + \frac{\gamma^2}{2})^2}{2\gamma^2}} dx \end{aligned} \quad (7)$$

By completing the squares in  $x$  and  $y$ , splitting the expression into the sum of two double integrals, and transforming one of the resulting outer integrals such that they are both integrated w.r.t. the same normal density function, it can be shown that the expression equals zero, independent of the value of  $K$ , if and only if  $\alpha$  is given by (3). Q.E.D.

Thus, the lower the variance of the noise, the more weight the option holder puts on the noisy observation at maturity relative to the (noiseless) initial price observation. As the variance of the noise goes from zero to infinity, the weight put on the initial price goes from 0 to 1. More precisely, we have the following proposition.

## Proposition 2

As  $\gamma$  approaches infinity the terminal payoff to the option is linear in the value of the asset.

### Proof

Since we know that  $S_\tau > 0$  a.s., we only need consider positive values of  $S_\tau$ . It then follows from (6) that the limit of the exercise probability will go to zero or one according to whether  $F_{0,\tau} < K$  or  $F_{0,\tau} > K$ . In the special case of  $F_{0,\tau} = K$ , an arbitrary convex combination of the two will be optimal, so we can set the exercise probability equal to zero in this case without loss of generality. In either case, the terminal payoff to the option is linear in  $S_\tau$ .

Q.E.D.

Another way to phrase Proposition 2 is that as the signal to noise ratio at maturity goes to zero the option can be treated as a forward contract for valuation purposes. As we will see later, the generalized version of this result for a finite noise variance is that the option can be treated as an option on a forward contract in the corresponding noiseless economy, where the time to maturity of the option is declining in the variance of the noise.

In the above proof, it is important to take the limit in the right order when considering what happens as the variance goes to infinity at the same time as the asset value goes to zero. If we fix  $\gamma$  and let the asset value go to zero, we obtain a

terminal payoff of 0, regardless of  $F_{0,\tau}$  and  $\gamma$ . Then, clearly the limit as  $\gamma$  goes to infinity must also be 0. But, we have just argued that in the case of  $F_{0,\tau} > K$  we will exercise with probability 1 in the limit as  $\gamma$  goes to infinity, for any  $S_\tau > 0$ . Thus, this must also hold in the limit, as  $S_\tau$  goes to 0, in which case the terminal payoff is  $-K$ . Therefore, we see that the conclusion depends on the order in which we take the limit. Since we know that the true asset value is strictly greater than zero with probability one, it would appear that the economically relevant limit is to fix any stock price greater than zero, and notice that the limit policy is to always exercise if the option is initially in the money.

Since the option is rationally exercised when  $P > K$ , the net value accruing to the option holder at maturity is given by:

$$V_\tau = (S_\tau - K) \chi_{\{P > K\}} \quad (8)$$

This gives rise to our next result, namely that the option no longer has a non-negative payoff in all states of the world.

### Proposition 3

The value at maturity of an ex ante rationally exercised call option will be negative with a strictly positive probability, given any combination of  $F_{0,\tau}$  and  $Z$  resulting in exercise, as long as  $\gamma > 0$ .

## Proof

Note that the value at maturity of the option is negative if  $S_\tau < K$ . All we need to show is therefore that  $P > K$  does not imply  $S_\tau > K$  with probability 1. Using (1) and the definition of  $Z$  it follows that we can write:

$$S_\tau = e^{-x} P^{\frac{1}{\alpha}} F_{0,\tau}^{\frac{(\alpha-1)}{\alpha}} e^{-\frac{\gamma^2}{2}} \quad (9)$$

That is, given the forecast,  $P$ ,  $S_\tau$  has a non-degenerate lognormal distribution. Thus, for any  $K > 0$ , the probability of  $S_\tau < K$  is strictly positive, since the lognormal distribution has all of  $\mathbb{R}_+$  as its support. Finally, note that the result is independent of the individual components of  $P$ .

Q.E.D.

For each value of  $S_\tau$ , we can compute the terminal value of the option at maturity given the outcome of the noise variable  $x$ . If we plot this terminal value function for a given  $x$ , as illustrated in Figure 1, we see that if  $x < 0$ , the function is discontinuous but non-negative and (weakly) monotonic. For  $x > 0$ , however, the function is neither non-negative nor monotonic. Only if  $x$  happens to be exactly equal to zero do we recover the standard continuous, non-negative, and weakly monotonic and convex payoff generally associated with call options.

We can take the expected value of the terminal value of the option for each value of  $S_\tau$  by integrating over the distribution of  $x$ . Under our parametric

assumptions, the probability of rational exercise for a given value of  $S_\tau$  can be shown to be given by:

$$Pr\{P>K|S_\tau\} = \Phi\left[\frac{\gamma}{\sigma^2\tau}\ln\left(\frac{F_{0,\tau}}{K}\right) + \frac{1}{\gamma}\ln\left(\frac{S_\tau}{K}\right)\right] \quad (10)$$

Since not exercising the option results in a zero value, the expected value of the option at maturity as a function of  $S_\tau$  is therefore given by:

$$V_\tau(S_\tau) = (S_\tau - K) \Phi\left[\frac{\gamma}{\sigma^2\tau}\ln\left(\frac{F_{0,\tau}}{K}\right) + \frac{1}{\gamma}\ln\left(\frac{S_\tau}{K}\right)\right] \quad (11)$$

This function is illustrated in Figure 2, and it has several interesting properties when compared to the standard noiseless call option payoff. First of all, if the final asset value happens to equal the exercise price, the exercise decision is irrelevant and thus the presence of the noise does not entail any loss in value. As the stock price approaches zero, the probability of exercising approaches zero, and since the loss from exercising is finite, the expected loss due to the noise also approaches zero. Similarly, as the stock price goes to infinity the probability of exercising approaches one and the expected value of the loss approaches zero since the loss from not exercising grows linearly in the stock price whereas the probability of such a loss declines exponentially. We also notice that the further the option is in-the-money (out-of-the-money) at time 0, the more likely will the holder be to exercise an out-of-

the-money option (not exercise an in-the-money option) at maturity for a given value of  $S_\tau$ . Since, for values of the terminal asset price less than  $K$  there is a strictly positive probability of values of  $x$  resulting in exercise, these asset values are associated with a negative expected terminal value of the option. This is a stronger result than Proposition 3, so we will record it as Proposition 4.

#### Proposition 4

For any value of the terminal asset price less than the exercise price, the expected terminal value of the option is strictly negative.

#### Proof

From the form of (11) it is trivially seen that  $S_\tau < K$  results in a non-positive value of  $V_\tau$ . In addition, as long as  $\gamma > 0$  it follows that the conditional probability of exercise is strictly positive for any values of  $S_\tau$  and  $K$ .

Q.E.D.

It also follows immediately from the discussion above that since the expected terminal value of the option goes to zero when the underlying asset value approaches zero and that the value is strictly negative for asset values between 0 and  $K$  that there must be a range of asset values wherein the expected option value declines in the value of the underlying asset. This result is somewhat non-intuitive and contrasts sharply with corresponding results for options under full observability of prices. The

intuition behind this result is that, even if the loss on an "accidentally" exercised out of the money option declines dollar for dollar as the true asset price increases, the probability of such an exercise occurring will grow faster than linearly for sufficiently low asset prices. Thus we have:

### Proposition 5

For  $0 < S_\tau < S^* < K$ , we have that the value at maturity of a rationally exercised call option is strictly declining in  $S_\tau$ , where  $S^*$  is the unique solution to the following equation in  $S$ :

$$\Phi\left[\frac{\gamma}{\sigma^2\tau}\ln(F_{0,\tau}/K) + \frac{1}{\gamma}\ln(S/K)\right] + \frac{1}{\gamma}\left[1 - \frac{K}{S}\right]\phi\left[\frac{\gamma}{\sigma^2\tau}\ln(F_{0,\tau}/K) + \frac{1}{\gamma}\ln(S/K)\right] = 0 \quad (12)$$

### Proof

By differentiating (11) with respect to  $S_\tau$  and setting the resulting expression equal to zero we have (12), which is the first order condition for a critical point of  $V_\tau$ . It suffices to show that such a critical point exists, is unique, and lies in  $(0, K)$  to prove the proposition, since in this case  $V_\tau$  will be below zero, and obviously not a maximum. This will be shown by rewriting (12) as:

$$S = G(S)$$

where

$$G(S) = \frac{K}{1 + \gamma \frac{\Phi[h(S)]}{\phi[h(S)]}} \quad (13)$$

$$h(S) = \frac{\gamma}{\sigma^2 \tau} \ln(F_{0,\tau}/K) + \frac{1}{\gamma} \ln(S/K)$$

The existence of a unique interior minimum of  $V_\tau()$  is thus equivalent to the existence of a unique fixed point for the function  $G()$ . We will note that as  $S$  ranges from  $0^+$  to  $\infty$ ,  $h(S)$  is continuous and strictly increasing from  $-\infty$  to  $\infty$ , and also use the following properties of the normal distribution:

- i)  $\Phi(x)/\phi(x)$  is strictly increasing in  $x$ ,
- ii)  $\lim_{x \rightarrow \infty} [\Phi(x)/\phi(x)] = \infty$ ,

and that

- iii)  $\lim_{x \rightarrow -\infty} [\Phi(x)/\phi(x)] = 0$ .

Then, it is easily seen that as  $S$  goes from  $0^+$  to  $\infty$ ,  $G(S)$  is continuous and strictly declining from  $K$  to  $0$ . Thus, by the mean value theorem,  $\exists! S^*$  satisfying  $0 < S^* < K$  and  $G(S^*) = S^*$ , and the proof is complete.

Q.E.D.

Before we compute the exact initial valuation formula for the option we record some general results on the relationship between the initial option values and the variance of the noise.

## Proposition 6

The initial value of an option is always strictly positive as long as  $\gamma$  is finite. The value of the option is always strictly lower than the corresponding Black-Scholes value as long as  $\gamma$  is positive. The intrinsic value of the option is independent of  $\gamma$ , so any effect on the option value of the noise at maturity enters through the time value.

## Proof

The value of the option is non-negative since the holder can always implement an exercise policy of never exercising, thus leaving it with a certain zero payoff and value. Also, if the option is initially in the money (comparing the exercise to the asset's forward price), then a policy of always exercising results in a terminal value equal to that of a forward contract which then has a positive initial value by assumption. Out of the money options have initial value if any information about the terminal value of the asset can be extracted from the noisy price report, such that a sufficiently high  $Z$  implies that the expected value of  $S_\tau$  is higher than  $K$ . This will be the case as long as the noise has finite variance. It can be seen from (6) that, as long as  $\gamma > 0$ , the probability of making (ex post) wrong exercise decisions is strictly positive, and thus the initial value must reflect these losses relative to the noiseless situation. The intrinsic value of the option,  $e^{-r\tau} \max[F_{0,\tau} - K, 0]$ , does not in any way involve the noise term,  $x$ .

Q.E.D.

### Proposition 7

The initial value of a European call option with time to maturity  $\tau$ , exercise price  $K$ , initial asset price  $S_0$ , asset return standard deviation  $\sigma$  per time period, with a risk free interest rate given by  $r$ , and with the variance of the noise in the observed terminal asset value is given by  $\gamma^2$ , is:

$$\begin{aligned}
 V_0 &= e^{-r\tau} \int_{-\infty}^{\infty} (e^y - K) \Phi \left[ \frac{\gamma}{\sigma^2 \tau} \ln \left( \frac{F_{0,\tau}}{K} \right) + \frac{1}{\gamma} (y - \ln(K)) \right] \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(y - \ln(F_{0,\tau}) + \frac{\sigma^2\tau}{2})^2}{2\sigma^2\tau}} dy \\
 &= S_0 \Phi \left[ \frac{\ln(F_{0,\tau}/K) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}} \right] - K e^{-r\tau} \Phi \left[ \frac{\ln(F_{0,\tau}/K) - \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}} \right] \\
 &\quad - e^{-r\tau} \int_{-\infty}^{\ln(K)} (K - e^y) \Phi \left[ \frac{\gamma}{\sigma^2 \tau} \ln \left( \frac{F_{0,\tau}}{K} \right) + \frac{1}{\gamma} (y - \ln(K)) \right] \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(y - \ln(F_{0,\tau}) + \frac{\sigma^2\tau}{2})^2}{2\sigma^2\tau}} dy \\
 &\quad - e^{-r\tau} \int_{\ln(K)}^{\infty} (e^y - K) \Phi \left[ \frac{\gamma}{\sigma^2 \tau} \ln \left( \frac{K}{F_{0,\tau}} \right) + \frac{1}{\gamma} (\ln(K) - y) \right] \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(y - \ln(F_{0,\tau}) + \frac{\sigma^2\tau}{2})^2}{2\sigma^2\tau}} dy
 \end{aligned} \tag{14}$$

### Proof

Writing  $S_\tau = e^y$  it follows that, under the risk adjusted probability measure,  $y$  is normally distributed with a mean of  $\ln(F_{0,\tau}) - \sigma^2\tau/2$  and a variance of  $\sigma^2\tau$ . This, together with (11) and our pricing assumption immediately imply the first equation in (14). The second equation follows from straightforward manipulation of the integral.

Q.E.D.

We recognize the first line in the option valuation formula as the Black-Scholes value of the option, i.e. the value the option would have had if there were no noise ( $\gamma=0$ ). The second and third line represent, respectively, the loss in value due to exercising the option when it is out of the money, and the loss arising from not exercising the option when it is in the money. It is easily seen that the integrands in each of these terms are strictly positive, as long as gamma is positive and finite.

The valuation formula given in (14) is somewhat cumbersome from both a numerical and analytical point of view, even though the intuition behind its derivation is very straightforward. Clearly, a very similar form can be derived by first conditioning on the asset price and integrating over  $x$ , and subsequently integrating over the asset price distribution. Unfortunately, the resulting valuation formula is equally complex and we do not believe it warrants further study. Instead, we will in the following derive two alternative characterizations of the initial value of the option, both of which are in the form as the Black-Scholes formula and therefore a much more convenient starting point for computations and comparative statics. Furthermore, we think that the derivations of these alternative formulas may have some independent interest in terms of shedding additional light on the source of the time value of options.

### Proposition 8

The initial value of the option given in (14) can also be represented as the Black-Scholes value of an option on the same asset with  $\tau$  replaced by  $\alpha\tau$  and  $K$

replaced by  $Ke^{-(1-\alpha)\tau r}$ , where  $\alpha$  is given by (3). That is, the value is:

$$V_0 = S_0 \Phi \left[ \frac{\ln(F_{0,\alpha\tau}/(Ke^{-(1-\alpha)\tau r})) + \frac{\sigma^2 \alpha \tau}{2}}{\sigma \sqrt{\alpha \tau}} \right] - Ke^{-(1-\alpha)\tau r} e^{-r\alpha\tau} \Phi \left[ \frac{\ln(F_{0,\alpha\tau}/(Ke^{-(1-\alpha)\tau r})) - \frac{\sigma^2 \alpha \tau}{2}}{\sigma \sqrt{\alpha \tau}} \right] \quad (15)$$

### Proof

Consider first the value of an option to enter into a long position in a forward contract on the same underlying asset in a standard noiseless Black-Scholes economy. The option matures at  $\alpha\tau$ , for some  $\alpha$  in  $(0,1)$ , its exercise price is given by  $K$ , and the forward contract matures at time  $\tau$ . Using standard results for the pricing of options on forward contracts, we find the value of this contract to be given by (15). Now, assume that if the option is exercised, the forward contract is held to maturity, at which time it will yield a net cash flow of  $S_\tau - K$ . Therefore, the total cash flow at  $K$  is given by:

$$W_\tau = (S_\tau - K) \chi_{\{F_{\alpha\tau,\tau} > K\}} \quad (16)$$

Compare (16) to (8), bearing in mind our assumption of pricing by taking expectations under a unique martingale measure and discounting at a constant risk free rate of interest. It then follows that the value of this forward option equals the value of the option that we want to price if the joint distribution of  $S_\tau$  and  $F_{\alpha\tau,\tau}$  under the equivalent martingale measure coincides with the joint distribution of  $S_\tau$  and  $P$ .

in the noisy world. It is straightforward to show that this is the case if and only if  $\alpha$  is given by (3).

Q.E.D.

### Proposition 9

The initial value of the option given in (14) can also be represented as the Black-Scholes value of an option on the same asset with  $\sigma^2$  replaced by  $\alpha\sigma^2$ , where  $\alpha$  is given by (3). Thus, the value can be expressed as:

$$V_0 = S_0 \Phi \left[ \frac{\ln\left(\frac{F_{0,\tau}}{K}\right) + \frac{(\alpha\sigma^2)\tau}{2}}{(\sqrt{\alpha}\sigma)\sqrt{\tau}} \right] - Ke^{-r\tau} \Phi \left[ \frac{\ln\left(\frac{F_{0,\tau}}{K}\right) - \frac{(\alpha\sigma^2)\tau}{2}}{(\sqrt{\alpha}\sigma)\sqrt{\tau}} \right] \quad (17)$$

We believe that it is instructive to give two independent proofs of this proposition. The first proof relies on an economically motivated argument, deriving the result from first principles. The second proof takes advantage of the previous proposition and a homogeneity property of the Black-Scholes model. That is, aside from a discounting effect, the value of the Black-Scholes formula remains constant under simultaneous changes of the time to maturity and the asset return variance that leave their product unchanged.

### Proof #1

Rewrite the payoff function given in (8) as:

$$\begin{aligned}
 V_\tau &= (P-K)\chi_{\{P>K\}} + (S_\tau - P)\chi_{\{P>K\}} \\
 &= \max\left[S_\tau^\alpha e^{\alpha x} F_{0,\tau}^{(1-\alpha)} e^{\frac{\alpha \gamma^2}{2}} - K, 0\right] + (S_\tau - P)\chi_{\{P>K\}}
 \end{aligned} \tag{18}$$

The first expression can be valued by the standard technique once we realize that under the equivalent martingale measure, the stochastic term has a log-normal distribution with mean equal to the initial forward price, and the variance of its logarithm is given by:

$$\begin{aligned}
 \text{Var}[\ln(S_\tau^\alpha e^{\alpha x})] &= \alpha^2 \sigma^2 \tau + \alpha^2 \gamma^2 \\
 &= \left(\frac{\sigma^2 \tau}{\sigma^2 \tau + \gamma^2}\right)^2 (\sigma^2 \tau + \gamma^2) \\
 &= \alpha \sigma^2 \tau
 \end{aligned} \tag{19}$$

Therefore, the time 0 value of the first term is given by the Black-Scholes formula with  $\sigma^2$  replaced by  $\alpha \sigma^2$ , and in order to complete the proof of the theorem we need to show that the present value of the second term is 0 for all values of K. But, as we demonstrated in the proof of Proposition 1, this term has an expected value of 0 under the risk adjusted probability measure for any  $K>0$ , and its present value is therefore 0.

Q.E.D.

## Proof #2

This proposition follows directly from the previous one by direct comparison of (15) and (17), using the following simple relationships:

$$\begin{aligned}
 (\alpha \sigma^2) \tau &= \sigma^2 (\alpha \tau) \\
 \sqrt{\alpha} \sigma \sqrt{\tau} &= \sigma \sqrt{\alpha \tau} \\
 e^{-r\tau} K &= e^{-r\alpha\tau} (K e^{-r(1-\alpha)\tau}) \\
 \ln\left(\frac{F_{0,\tau}}{K}\right) &= \ln\left(\frac{S_0 e^{r\tau}}{K}\right) = \ln\left(\frac{F_{0,\alpha\tau}}{K e^{-r(1-\alpha)\tau}}\right)
 \end{aligned} \tag{20}$$

Q.E.D.

In order to directly confirm our previous claim that only the time value of the option is affected by the unobservable nature of the asset's final value, we can rewrite the two Black-Scholes formula representations above in terms of the intrinsic value/time value decomposition of Carr and Jarrow (1990):

$$\begin{aligned}
 V_0 &= \max[S_0 - K e^{-r\tau}, 0] + e^{-r\tau} \frac{(\sqrt{\alpha} \sigma) K}{2} \int_0^\tau \frac{1}{\sqrt{t}} \phi\left(\frac{\ln\left(\frac{F_{0,\tau}}{K}\right) + \frac{(\alpha \sigma^2) t}{2}}{(\sqrt{\alpha} \sigma) \sqrt{t}}\right) dt \\
 &= \max[S_0 - K e^{-r\tau}, 0] + e^{-r\alpha\tau} \frac{\sigma K e^{-r(1-\alpha)\tau}}{2} \int_0^{\alpha\tau} \frac{1}{\sqrt{t}} \phi\left(\frac{\ln\left(\frac{F_{0,\alpha\tau}}{K e^{-r(1-\alpha)\tau}}\right) + \frac{\sigma^2 t}{2}}{\sigma \sqrt{t}}\right) dt
 \end{aligned} \tag{21}$$

The conclusion is that the option pricing formula in the case of the noisy signal

at maturity can be treated as the noiseless Black-Scholes formula with the assets return volatility replaced with a measure of its average "revealed volatility". That is, the higher is the noise in the final observation, the less of the asset price volatility is actually revealed, and it is from the revelation of asset price volatility that the option derives its time value, not from the volatility itself.

## II. TWO APPLICATIONS

In order to demonstrate the basic model developed above, we will give two examples of applications of the framework. The first relates to Merton's model of the risk structure of corporate debt, and the second to an option based valuation approach to research and development projects.

### II.1 Pricing Risky Corporate Debt

Consider an upstart company that is raising  $S_0$  in debt and equity. The debt is in the form of a zero coupon bond with face value of  $K$  that matures in  $\tau$  years from now. The bond covenants preclude the company from making any dividend payouts before  $\tau$ , and it is also assumed that the capital budgeting decisions of the firm are fixed in such a way that the value of the firm's assets from time 0 to time  $\tau$  follows a geometric Brownian Motion with a standard deviation of  $\sigma$  per year. Finally, we assume that the value of the firm's assets at maturity is unobservable, but

that an appraisal will be made, resulting in an estimated value of the firm equal to  $S_\tau e^x$ , where  $x$  is distributed as  $N(-\gamma^2/2, \gamma^2)$ . We would argue that this is a more realistic scenario than one where the company management/owners can exactly observe the true value of the assets at the maturity time, as assumed in previous applications of the option pricing methodology to this setting. We will maintain the assumptions necessary to price the corporate liabilities through the equilibrium martingale measure, and ask the question: What is the equilibrium promised yield to maturity on the company's risky debt?

As is well known, the initial value of the equity,  $V_0$ , should be exactly equal to the value of a maturity  $\tau$  European call option on the assets of the company with exercise price  $K$ , and the initial value of the debt is therefore  $S_0 - V_0$ . We have already valued such an option in Propositions 7-9, and there we concluded that the value of the option is equal to its Black-Scholes value minus two positive terms caused by the unobservability of the asset value at maturity. Thus, ignoring this unobservability would tend to overstate the initial value of the equity and correspondingly underestimate the value of the debt. This would result in a promised yield to maturity of the debt that is too high. Notice that this result is not due to any informational asymmetries; the debtholders' knowledge at maturity is completely irrelevant, since they are not making any decisions at this time.

In order to gauge the economic importance of the unobservable asset values at maturity, we use (17) to compute the equilibrium promised return on the bonds for different values of  $\gamma$  and  $K$ , in a setting where the other parameters were held

constant as  $S_0=1000$ ,  $\sigma=.3$ ,  $r=10\%$ , and  $\tau=1$ . The equilibrium promised return is computed as  $\ln(K/(S_0-V_0))$ , where  $V_0$  is given by (11). Table 1 reports the results for some plausible (?) combinations of the face value of the debt and noise at maturity.

**Table 1 - EQUILIBRIUM PROMISED YIELD (in % p.a.)**

	K=700	K=800	K=900
$\gamma^2=0.00$	11.05	12.54	14.97
$\gamma^2=0.01$	10.82	12.15	14.41
$\gamma^2=0.02$	10.65	11.82	13.95
$\gamma^2=0.03$	10.52	11.57	13.56

The first row in the table gives the equilibrium promised yields to maturity on the risky bonds in the case of a fully observable asset value at maturity, i.e. the standard solution. The following three rows contain the corresponding yields for increasing, but seemingly low, levels of variance for the log of the noise at maturity. We find it interesting that such low levels of noise can have so dramatic impacts on the bond yields, and we take this as an indication that our model represents a significant economic issue, as opposed to a theoretical curiosity.

## II.2 Valuing a R&D Limited Partnership

Our second example will address the question of the *ex ante* value of an information system in connection with an investment in a risky project. Consider a company that is evaluating a capital budgeting proposal to invest in a research and development limited partnership<sup>4</sup> with a fixed time horizon of two years. At the end of this time period, the company must decide whether to drop the project or buy out the other partners at a cost of  $K$  dollars. In the former case, only the initial investment is lost, whereas in the latter case, the firm achieves full rights to commercial use of the research generated by the partnership. If we denote the value of these rights after 2 years as  $S_2$  and assume that this value is lognormally distributed and unobservable, it is clear how we can apply our framework to determine a fair value of a partnership investment. We believe that it is quite reasonable that the uncertainty about the value of the research after 2 years is unobservable, and that the measurement error resulting from an appraisal can be substantial. Also, it is more likely than not that the "option" is initially out of the money, i.e. that if the company had to make a commitment decision at time zero they would have rejected it. Otherwise, it would seem that keeping the project in house would be a more likely arrangement than farming it out to a partnership.

Our additional twist on this problem is that if the company makes the

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<sup>4</sup> For a recent example of applications of traditional option pricing methods to value such partnership, see Shevlin (1991). He actually assumes that  $K$  is random as well and uses Margrabe's (1978) exchange option pricing model for his valuation purposes, but this is not an important distinction.

investment, they can also invest  $J$  (time 0) dollars in managerial capacity<sup>5</sup>, and that the more they invest the more precise information they will receive about  $S_2$ . To be specific, we will assume that the variance of the noise in the appraisal at maturity is given by:

$$\gamma^2 = \gamma_0^2 e^{-cJ} \quad (22)$$

Here,  $\gamma_0$  is the standard deviation of the noise when no new capacity is added, whereas the more (in terms of quantity and/or quality) managerial capacity is added, the lower will  $\gamma$  be. The parameter  $c$  is a measure of the cost efficiency of the available information system technology. The company's decision problem can now be solved by first valuing the option for a given  $J$  (and hence  $\gamma^2$ ), and then maximizing the resulting value over  $J$ . From our previous analysis it follows directly that the initial value of the project as a function of  $J$  is given by:

$$V_0(J) = S_0 \Phi \left[ \frac{\ln\left(\frac{F_{0,\tau}}{K}\right) + \frac{\alpha(J)\sigma^2\tau}{2}}{\sqrt{\alpha(J)\sigma\sqrt{\tau}}} \right] - Ke^{-r\tau} \Phi \left[ \frac{\ln\left(\frac{F_{0,\tau}}{K}\right) - \frac{\alpha(J)\sigma^2\tau}{2}}{\sqrt{\alpha(J)\sigma\sqrt{\tau}}} \right] - J \quad (23)$$

where

$$\alpha(J) = \frac{\sigma^2\tau}{\sigma^2\tau + \gamma_0^2 e^{-cJ}}$$

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<sup>5</sup> We use the term "managerial capacity" as an all-inclusive term for investments in people and/or information systems that are irreversible in the short run and that will enhance the company's ability to observe and evaluate the project.

This function can be shown to have a unique maximum (not necessarily interior), which can, in principle, be found by standard calculus tools. However, since we need to solve a non-linear equation to find the optimal  $J$ , we will just present numerical results here. To be specific, assume that  $\sigma=.4$ ,  $S_0=1000$ ,  $K=1500$ ,  $r=10\%$ , and  $\gamma_0^2=.2$ . The results are given in Table 2 for a wide range of values of  $c$ .

**Table 2 - OPTIMAL DECISIONS AND VALUES**

$c$	$J$	$\gamma^2$	$V(J)$
<.0303	0	.200	101.75
.0303	0.15	.199	101.75
.0305	0.71	.196	101.75
.031	1.99	.188	101.77
.032	4.24	.175	101.87
.033	6.15	.163	102.03
.034	7.80	.153	102.24
.035	9.23	.145	102.49
.036	10.48	.137	102.77
.037	11.56	.130	103.07
.038	12.52	.124	103.39
.039	13.37	.119	103.73
.04	14.12	.114	104.08
.05	18.25	.080	107.76
.06	19.43	.062	111.22
.07	19.52	.051	114.23
.08	19.16	.043	116.81
.09	18.61	.037	119.04
.1	18.00	.033	120.97
.2	12.86	.015	131.70
.3	10.01	.010	136.33
.4	8.25	.007	138.95
.5	7.06	.006	140.65
1	4.24	.003	144.50
2	2.47	.001	146.77
5	1.17	.001	148.37

It turns out that if the managerial cost efficiency is sufficiently low ( $c < .0303$ ), the optimal solution is not to add any managerial capacity, and the value of the project is equal to the base case value of 101.75 (\$ mill.). However, for slightly higher values of  $c$  the company will invest a considerable amount of money on an information system. At first this will not generate significantly higher net values of the project, but as the cost efficiency improves, the resulting increase in project value from an optimal information system investment gets to be substantial. If there are technological solutions available that allow for a virtual elimination of the noise at maturity at a trivial cost, the net value of the project is nearly 50% higher than in the base case. Note that the optimal amount of money spent on information system capacity is first increasing and subsequently decreasing in  $c$ . These results are illustrated in Figure 3.<sup>6</sup>

### III. CONCLUSIONS AND SOME GENERALIZATIONS OF THE MODEL

We have considered the valuation of European call options<sup>7</sup> in a setting which differs from previous analyses primarily by admitting the possibility that the holder of the option is restricted from gaining perfect information about the value of the underlying asset at maturity. By choosing a convenient parametric form for the noise in the signal of this underlying value, we were able to derive exact quasi-closed form

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<sup>6</sup> One might find it interesting to compare this result with the pattern of investment in computer hardware, as its cost efficiency has dramatically increased over the last 30 years.

<sup>7</sup> The valuation of European puts follows directly from put-call parity in this model.

solutions to the optimal exercise and valuation problems. These solutions allowed us both to analyze the problem further in theory and to work out a couple of illustrating examples, hopefully demonstrating that the situation we have modeled is interesting and economically relevant.

Having said this, it is clear that if we want to apply this type of model to real world problems it would be useful to make some theoretical generalizations. First of all, we should consider relaxing the "asymmetric" assumption of the intial value of the underlying asset being observable while the future value is unobservable. As long as we model the initial noise as being of the same form as that at maturity, this does not seriously complicate the computations. However, it does bring up questions of information asymmetries. Unless the initial, noisy signal of the asset value is publicly observable, we have to address issues of private information aggregation and the potential for a winner's curse problem.

In addition, we would also like to be able to address the valuation of exchange options (Fischer, 1978; Margrabe 1978) with noisy signals of both the underlying asset value and the exercise price, as well as for compound options (Geske 1977, 1979) and American type claims that can be exercised before maturity. The latter two problems open up the question of how to model multiple signals about the underlying asset value. These could come continuously if we can observe trading of a marketed asset whose price is imperfectly correlated with the underlying asset value, or they could come at random times and with random variance, e.g. at times when other assets of varying degrees of similarity are being sold. Finally, we should allow for the

possibility of the holder of the option seeking further information, depending on the signals received<sup>8</sup>.

We believe that most of these issues can be fruitfully addressed, although some of them may result in serious problems of tractability. The same holds for attempts to generalize the distributional assumptions of the noise term. Of particular interest in this regard will be signals of the form  $S_t > L$ , for some limit value  $L$ . These type of signals can arise in connection with markets that are subject to trading halts when prices moves too far too fast, and where some market participants possess options of one form or another. A direct example of this problem would be found in the valuation of futures options when there is a possibility of the futures contract having limited out on the maturity day of the option. This is also closely related to the work by Brennan (1986) on the relationship between futures price limits and the decisions of traders to default on margin calls.

These questions, along with a host of applications are the subject of ongoing research. Hopefully, we will soon be able to report more progress.

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<sup>8</sup> Comparing to the R&D example, seeking such additional information only if ex post desired would correspond to relying on external "consultants" instead of building up an internal managerial capacity.

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FIGURE 1  
OPTION VALUE AT MATURITY  
FOR DIFFERENT VALUES OF  $X$

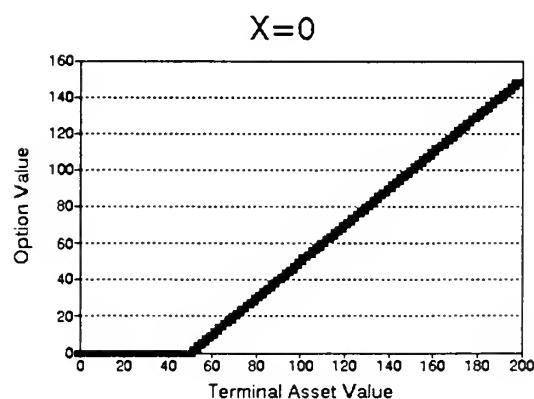
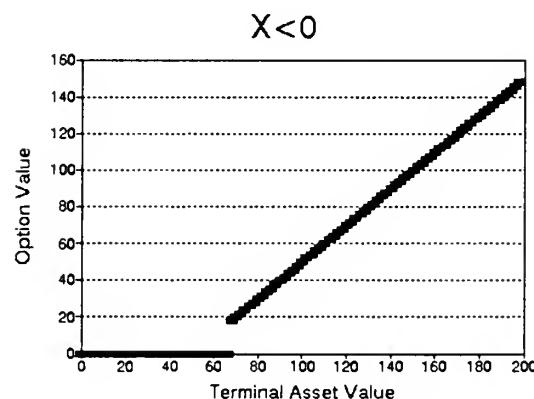
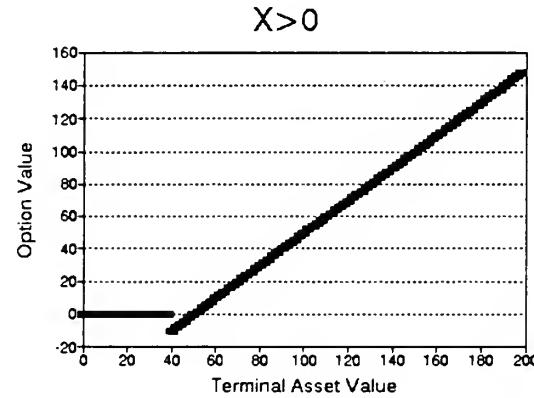


FIGURE 2

## TERMINAL OPTION VALUES FOR UNOBSERVABLE TERMINAL ASSET VALUE

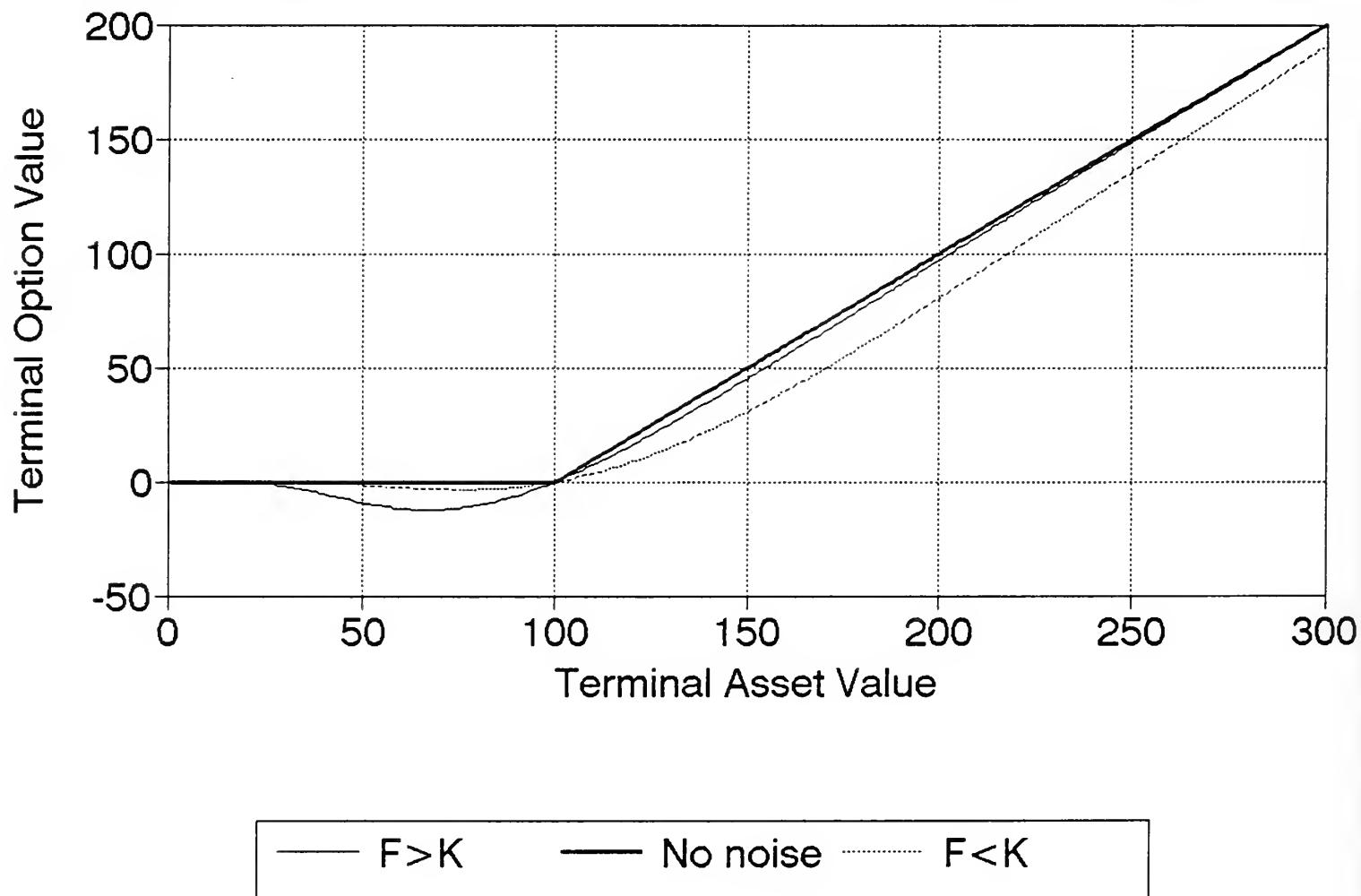
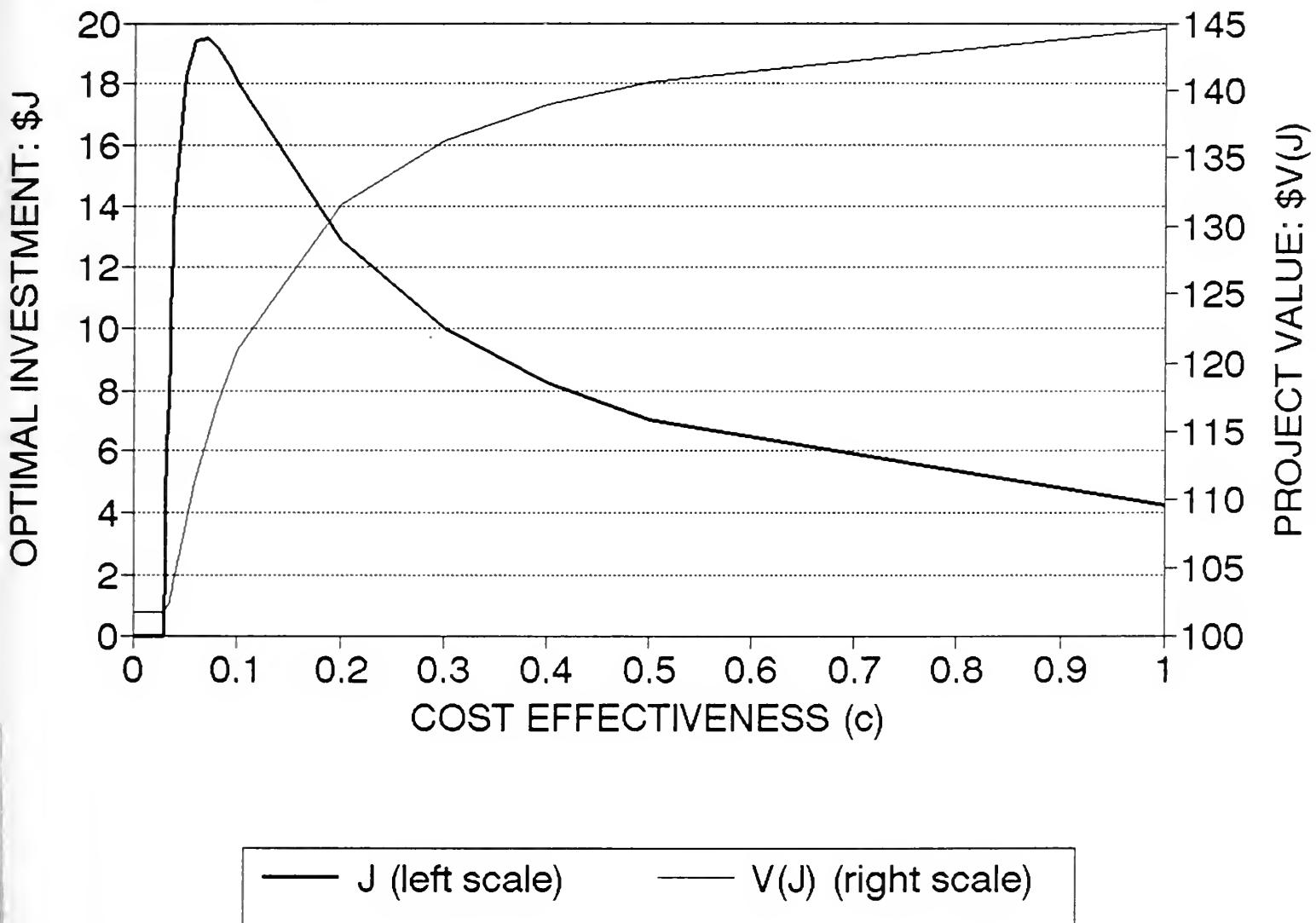


FIGURE 3

OPTIMAL DECISIONS AND PROJECT VALUE  
AS FUNCTIONS OF COST EFFECTIVENESS







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